

Entanglement and Irreversibility in the Approach to Thermal Equilibrium

Known and New Results on Thermalizing Quantum Channels for Qubits

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Abstract. When a physical system is put in contact with a very large thermal bath, it undergoes a dissipative (i.e., an apparently irreversible) process that leads to thermal equilibrium. This dynamical process can be described fully within quantum physics, involving only unitary, therefore reversible, maps. The information, initially present in the system, is not erased, but is diluted in the bath because of entanglement. Irreversibility may arise if, after quantum information has been thus diluted, some classical information is lost. This paper reviews a model for thermalization that displays these features. Two new analytical results are provided for the zero-temperature channels: a new quantitative measure of entanglement, and a study of irreversibility in the case where the lost classical information is the label of the particles in the bath.

1 Introduction

A well-established tenet of statistical physics says that an ensemble of physical systems in thermal equilibrium with a large reservoir (“canonical ensemble”) will show a statistical behavior: a state of energy E will be occupied with a probability proportional to $e^{-\beta E}$ where $\beta = \frac{1}{k_B T}$, T is the temperature and k_B is Boltzmann’s constant. Statistical physics, whose fruitfulness is beyond question, takes the existence of statistical ensembles as a starting point. The question about the origin, and even the meaning, of these statistics, is ultimately still unsettled. Recent remarkable developments have shown that ensembles are somehow generic in a *kinematic* sense: for instance, if one picks at random a state $|\Psi\rangle$ in the Hilbert space describing a large number N of particles, then, under suitable constraints, the state of much smaller sub-systems $\rho_n = \text{Tr}_{N-n}|\Psi\rangle\langle\Psi|$ shall almost always be close to a canonical state [1,2,3,4,5].

Here, we consider rather a *dynamical* process, the approach to thermal equilibrium, or *thermalization*, a typical example of dissipative phenomenon¹. It is defined as follows. A large number N of particles are already in the canonical state ξ ; one brings along a new particle, prepared in an arbitrary state ρ , possibly pure: thermalization is the process, at the end of which the new particle reaches arbitrarily close to the canonical state, while the bath is almost

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¹ Some terminological issues to avoid confusion: I use *dissipation* as the phenomenological fact that energy, or information, which was initially concentrated in some physical system, flows through the evolution into the systems that interact with the first one. When a great number of degrees of freedom is involved, dissipation leads to apparent (and practical) *irreversibility*; but contemporary physics does not contemplate any fundamentally irreversible process.

unmodified. The naive description of this process,

$$\rho \otimes \xi^{\otimes N} \longrightarrow \xi^{\otimes(N+1)}, \quad (1)$$

is clearly non-unitary, because input orthogonal states are mapped on the same final state. For sure, non-unitary maps are allowed in quantum physics as descriptions of open systems: specifically, (1) is possible if the bath itself is coupled to an environment, to which it transfers all the information about ρ . Remarkably, quantum physics allows to find *unitary* maps that involve only the bath and the new particle, and that describe thermalization as well. This possibility is due to *entanglement*: the information, which was initially concentrated in ρ , is not lost in an external environment, but is encoded differently, being spread between the system and the bath in a coherent and reversible way. As a result, one may have

$$\rho \otimes \xi^{\otimes N} \longrightarrow \sigma \text{ such that } \text{Tr}_N \sigma \approx \xi. \quad (2)$$

This means that the final state is not the exact thermal state of $N + 1$ particles $\xi^{\otimes(N+1)}$; however, when one picks any particle out, its partial state is as close as desired to the thermal state. In other words, $\text{Tr}(\hat{A}\sigma)$ will correspond to the expected thermal average for all single-particle observables $\hat{A} = \frac{1}{N+1} \sum_{k=1}^{N+1} A_k$, with A_k the operator acting as A on particle k and trivially on the others. In short: entanglement allows to construct a dissipative channel with fully reversible dynamics. The study of such channels is richer, and probably more satisfactory from the standpoint of physics, than the simple acceptance of the naive map (1).

In this paper, I review the thermalizing channels that were presented a few years ago for a specific model of the system and the bath [6]. This approach to thermalization, that can be seen in the broader context of “quantum homogenization” [7,8], was proposed as a benchmark for exploring a quantitative link between entanglement and dissipation. I also present two new results for this model: a new measure of entanglement, and an analytical computation for a numerical result obtained in [7] for a model of irreversibility.

2 The Model and the Results

2.1 Definition of the Model

The model is defined by the following assumptions on the kinematics and dynamics.

Kinematics and Free Dynamics. The particle to be thermalized (the “system” S hereafter) is a two-level system, i.e. a qubit. The thermal bath is a reservoir composed of an arbitrary large number N of qubits, which interact with an external field but not with one another (a model featuring such interactions has been studied in Ref. [9]). The free Hamiltonian for the bath is therefore $H_B = \sum_{i=1}^N h[i]$ where $h[k]$ is the operator acting as $h = -E\sigma_z$ on the qubit k and trivially on the other qubits. We denote the projectors on the eigenstates of σ_z by $P_0 = |0\rangle\langle 0|$ and $P_1 = |1\rangle\langle 1|$. The initial state of the system is arbitrary. The single-particle equilibrium state is

$$\xi = e^{-\beta h} / \text{Tr}(e^{-\beta h}) = pP_0 + qP_1 \quad (3)$$

with $p = \frac{1}{2}(1 + \tanh(\beta E))$ and $q = 1 - p$. We set $E > 0$, so that $|0\rangle$ is the ground state, and $p = 1$ corresponds to $T = 0$. The initial state of the bath is the thermal state $\rho_B = e^{-\beta H_B} / \text{Tr}(e^{-\beta H_B}) = (\xi)^{\otimes N}$.

Interaction System-Bath. We consider a *collision model*, in which the system interacts sequentially with the qubits in the bath one by one. Each step of this stroboscopic evolution is described by a hamiltonian H , or the corresponding evolution $U = e^{iH}$, acting on $\mathbb{C}^2 \otimes \mathbb{C}^2$. Moreover, we consider that a qubit of the bath undergoes at most one interaction with the system (Fig. 1), so the input state of the ancilla is always ξ . The evolution of the system is finally described by the iteration of the completely-positive (CP) map T_ξ defined as

$$\rho^{(k+1)} = \text{Tr}_B [U (\rho^{(k)} \otimes \xi) U^\dagger] \equiv T_\xi [\rho^{(k)}]. \quad (4)$$

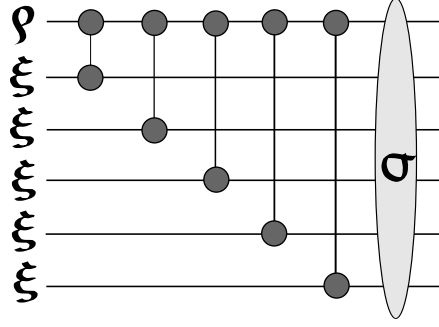


Fig. 1. The collisional model for thermalization: quantum circuit representation. The input (left) shows system qubit in an arbitrary state ρ , bath qubits in the thermal state ξ . The unitary operations connect sequentially the system qubit with each of the qubits in the bath. At the output (right), the qubits are in a correlated state σ . Thermalization is achieved if $\text{Tr}_N \sigma \approx \xi$ holds for all the single-qubit states, whatever the initial state ρ of the system qubit.

2.2 Requirements for a Thermalizing Channel

We want to find all the two-qubit unitary operations that define a thermalizing channel in our scenario. Four requirements are suggested by physics:

First Requirement. U should not depend on temperature (i.e. on p), but may depend on the label z of the local free Hamiltonian. The requirement is motivated by the fact that z is a locally available information, while temperature is a characteristic of an ensemble. Also, we want to characterize interactions that thermalize at all temperatures, avoiding possible pathological examples which would define the correct physics only for a specific value of this parameter. So from now on we write U_z .

Second Requirement. If the system is prepared in the local equilibrium state ξ , nothing should happen, because the thermalization is already achieved. Formally:

$$U_z (\xi \otimes \xi) U_z^\dagger = \xi \otimes \xi \quad \text{for } \xi = pP_0 + qP_1. \quad (5)$$

Note that this is stronger than requiring ξ to be a fixed point of T_ξ , because we ask that the qubit of the bath is unchanged too. Because of the first requirement, (5) should hold for all p .

Third Requirement. For any input state ρ of the system, the iteration of the map T_ξ leads to thermalization:

$$\rho^{(n)} = T_\xi^n[\rho] \longrightarrow \xi \quad \forall \rho. \quad (6)$$

Fourth Requirement. For any input state ρ of the system, the fluctuations introduced in the state of the bath are small.

2.3 Solution: All Thermalizing Channels for the Model

The combination of the first and the second requirements imply that the subspaces $P_0 \otimes P_0$, $P_1 \otimes P_1$ and $P_0 \otimes P_1 + P_1 \otimes P_0$ must be invariant under the action of U_z . To prove this assertion, we notice that on the l.h.s. of (5) the term $U_z P_0 \otimes P_0 U_z^\dagger$ appears with the weight p^2 , the term $U_z (P_0 \otimes P_1 + P_1 \otimes P_0) U_z^\dagger$ with the weight $p(1-p)$, and the term $U_z P_1 \otimes P_1 U_z^\dagger$ with the weight $(1-p)^2$. Since we want condition (5) to hold for all p , the three subspaces must be separately invariant. Thus the first and the second requirement restrict U to take the form

$$\begin{aligned} |00\rangle &\longrightarrow e^{i\chi_0} |00\rangle \\ |11\rangle &\longrightarrow e^{i\chi_1} |11\rangle \\ |01\rangle &\longrightarrow e^{i\chi_2} [\cos \phi |01\rangle + e^{i\varphi_2} \sin \phi |10\rangle] \\ |10\rangle &\longrightarrow e^{i\chi_3} [\cos \phi |10\rangle - e^{-i\varphi_2} \sin \phi |01\rangle] . \end{aligned} \quad (7)$$

Remarkably, each of these unitaries defines a thermalizing channel (in other words, the third requirement is automatically fulfilled when enforcing the first and the second). To prove this assertion, we compute explicitly $\rho^{(n)} = T_\xi^n[\rho]$ as a function of the parameters of the initial state ρ . Let's write

$$\rho^{(n)} = d^{(n)} P_0 + (1 - d^{(n)}) P_1 + k^{(n)} |0\rangle\langle 1| + k^{(n)*} |1\rangle\langle 0|. \quad (8)$$

Inserting the explicit form (7) for U_z into (4), we find that the effect of the map T_ξ is given by $d^{(n+1)} = d^{(n)} \cos^2 \phi + p \sin^2 \phi$ and $k^{(n+1)} = \cos \phi \lambda k^{(n)}$ with $\lambda = p e^{i(\chi_0 - \chi_3)} + q e^{i(\chi_2 - \chi_1)}$ (note that $|\lambda| \leq 1$). A straightforward iteration gives $d^{(n)}$ and $k^{(n)}$ as a function of the parameters $d^{(0)}$ and $k^{(0)}$ of the initial state ρ :

$$d^{(n)} = [1 - (\cos \phi)^{2n}] p + (\cos \phi)^{2n} d^{(0)}, \quad (9)$$

$$k^{(n)} = k^{(0)} (\lambda \cos \phi)^n. \quad (10)$$

Thus, whenever $\phi \neq 0$, the iteration of T_ξ yields $d^{(n)} \rightarrow p$ and $k^{(n)} \rightarrow 0$, i.e. thermalization (6). Finally, the fourth requirement is fulfilled by taking ϕ small enough. In fact, it can be verified that the fidelity of the state σ of a bath qubit after the interaction with respect to the thermal state satisfies $F = \text{Tr} \left[\left(\xi^{\frac{1}{2}} \sigma \xi^{\frac{1}{2}} \right)^{\frac{1}{2}} \right] \geq \cos \phi$ for all temperatures and for all input states of the system.

Of the six free parameters defining the most general U_z (7), only ϕ and the differences $\chi_0 - \chi_3$ and $\chi_2 - \chi_1$ define the CP-map T_ξ . The reason is that there are two symmetries of the physical process of thermalization. The first one is the usual choice of a global phase; the second is the freedom of choosing the global phases of $|0\rangle$ and $|1\rangle$ for qubits in the bath, both before and after the interaction with the system. Mathematically, if U_z defines a CP-map T_ξ , then, writing $u(x) = P_0 + e^{ix} P_1$ a rotation along the z axis, $e^{i\chi} [\mathbb{1} \otimes u(\alpha)] U_z [\mathbb{1} \otimes u(\beta)]$ defines the same CP-map for all choices of α and β . So much for the three parameters which play strictly no role. In addition, the phase of λ is a measurable parameter but does not add any insight on the thermalization process. In fact, this parameter is associated to a rotation around the z axis, or in other words, to a redefinition of the x and y axes of the Bloch sphere in the plane perpendicular to z , for the system qubit. Since there is nothing, in the physics of the model, that would single out a specific axis in the (x, y) plane, we can study thermalization for any specific choice of the phase of λ .

All in all, the physics of thermalization in this model can be studied on a two-parameter family of unitary transformations. A convenient choice is²

$$V_z(\phi, \theta) = \begin{cases} |00\rangle \longrightarrow e^{i\theta} |00\rangle \\ |11\rangle \longrightarrow e^{i\theta} |11\rangle \\ |01\rangle \longrightarrow \cos \phi |01\rangle + i \sin \phi |10\rangle \\ |10\rangle \longrightarrow \cos \phi |10\rangle + i \sin \phi |01\rangle \end{cases} \quad (11)$$

which can be written as $e^{i\theta/2} e^{iH(\phi, \theta)}$ with

$$H(\phi, \theta) = \frac{1}{2} [\phi (\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y) + \theta \sigma_z \otimes \sigma_z]. \quad (12)$$

2.4 The Physics of Thermalizing Channels

In spite of the extreme simplicity of the model, the thermalizing channels display a rich physics. Here is a review of these features. For the known results, the demonstration is given in Refs [6,7]; the new results will be demonstrated in Section 3 below.

² Here the sign of θ is the opposite than in Ref. [6].

The partial swap. The particular choice $\theta = \phi$ makes $V_z(\phi, \phi) \equiv V(\phi)$ independent of z : this is the only interaction, in this model, that is independent not only of temperature, but also on the parameter of the local Hamiltonian. This unitary can be written as

$$V(\phi) = \cos \phi \mathbb{1} + i \sin \phi U_{\text{swap}} \quad (13)$$

where $U_{\text{swap}} = V(\pi/2)$ is the operation that swaps the state of the two qubits: $|\psi_1\rangle|\psi_2\rangle \rightarrow |\psi_2\rangle|\psi_1\rangle$. The mechanism for thermalization is quite intuitive in this case. With a channel made of pure swaps, the system would be in state ξ after one single interaction, but this would introduce a large fluctuation in the bath. The partial swap, in the meaningful limit $\cos \phi \approx 1$, is the gentle version of this process: after many steps, the state of the system has been “swapped” into the bath, but the fluctuations in the bath itself are small.

Dissipation and Decoherence. The dynamics of thermalization is described in (9) and (10), with $\lambda = pe^{i\theta} + qe^{-i\theta}$. The parameter θ appears only in the dynamics of the off-diagonal term, hence is entirely related to decoherence. The re-equilibration of populations, i.e. dissipation, is governed by ϕ alone³. Interestingly, dissipation and decoherence are decoupled in this model, in the sense that $V_z(\phi, \theta) = V_z(\phi, 0)V_z(0, \theta) = V_z(0, \theta)V_z(\phi, 0)$. Also, one can verify that

$$V_z(\phi, \theta) = V_z(0, \theta - \phi)V(\phi) : \quad (14)$$

all the thermalizing unitaries in this model can be seen as the application of the partial swap followed by an additional reduction of coherence.

Continuous-time limit. In order to pass from the discrete dynamics indexed by n to a continuous-time dynamics with parameter t , one sets $n = t/\tau_0$, and lets the interaction time τ_0 go to zero together with ϕ and θ , keeping constant the dissipation rate $\frac{\phi^2}{\tau_0} = \frac{1}{T_1}$ and the phase fluctuation rate $\frac{2\theta^2}{\tau_0} = \frac{1}{T_{pf}}$. One finds that in the continuous-time limit, the processes of dissipation (9) and decoherence (10) are exactly exponential:

$$d(t) = e^{-t/T_1}d(0) + (1 - e^{-t/T_1})p, \quad (15)$$

$$|k|(t) = e^{-t/T_2}|k|(0) \quad (16)$$

with $\frac{1}{T_2} = \frac{1}{2T_1} + pq\frac{1}{T_{pf}}$. For $\theta = 0$ or at zero temperature, the bound $T_1 \geq \frac{1}{2}T_2$ (see e.g. [10], p. 120) is saturated. Note also that it is possible to relate this dynamics to a master equation of the Lindblad type [8].

Dissipation and Entanglement. As we mentioned in the introduction, this model was proposed as a benchmark for studying the link between entanglement and dissipation. This link has been explored along several directions. An especially strong link is provided by the following observation: we have seen that any thermalizing unitary is equivalent to one of the $V(\phi, \theta)$ up to local unitaries (LU), and it can be shown that all the $V(\phi, \theta)$ are inequivalent under LU [6,11]. Thus ϕ and θ (or any equivalent choice of two parameters) are necessary and sufficient to define all the properties of entanglement; but in turn, these parameters uniquely define the relaxation times T_1 and T_2 . So, for the model under study, the relaxation times are directly related to entanglement.

In Refs [6,7,12] several kinds of entanglement have been computed: the largest amount of two-qubit entanglement that a single application of $V(\phi, \theta)$ can generate, the entanglement of a given qubit versus all the others etc. These are measures of bipartite entanglement, the only ones which were available at the moment of writing. Computable measures of *multipartite* entanglement have been proposed since [13,14,15]. According to one of these measures, after interaction of the system with n qubits in the bath, an amount of entanglement⁴

$$\mathcal{E}_{n+1} = 2|c_1| \sqrt{(1 - c^{2n}) \left(1 - |c_1|^2 \frac{1 - c^{2(n+1)}}{1 + c^2}\right)}, \quad (17)$$

³ Note that ϕ has to appear in both the diagonal and off-diagonal term, as is indeed the case: any re-equilibration of populations must be balanced by a sufficient amount of variation in the coherence, in order to avoid evolution into non-physical states (matrices with negative eigenvalues).

⁴ For comparison, the same measure of entanglement applied to the $(n+1)$ -qubit GHZ state gives $\mathcal{E}_{n+1}(\text{GHZ}) = \sqrt{2}\sqrt{1 - 2^{-n}}$.

with $c = \cos \phi$, is generated at $T = 0$ for an arbitrary pure initial state $c_0|0\rangle + c_1|1\rangle$ (subsection 3.1). This expression is monotonically increasing in n ; for any given n , the maximal amount of entanglement is generated by the input state $|1\rangle$, which is indeed the farthest from equilibrium. For this case $c_1 = 1$, $\mathcal{E}_\infty = 2\frac{c}{\sqrt{1+c^2}} \rightarrow \sqrt{2}$ in the limit of weak interactions.

Irreversibility. The thermalization process that we described is certainly reversible, since it is described by the unitary operation $\mathcal{U} = U_{S,n} \dots U_{S,2} U_{S,1}$. To study irreversibility, one must add some lack of knowledge. In our context, the most natural way of doing this is to suppose that the labels of the qubits in the bath are randomly permuted before reversing the evolution \mathcal{U} , and we don't control this permutation. Then the reversed evolution reads $\mathcal{U}_\pi = \mathcal{U}^\dagger \Pi \mathcal{U}$ which is equal to the identity if and only if $\Pi = \mathbb{1}$. Note that Π is unitary, and consequently so is \mathcal{U}_π : apparent irreversibility is due to the fact that we reverse only \mathcal{U} , because we are not supposed to control Π . In Ref. [7], we have provided numerical results for this effect. In subsection 3.2 below, we compute analytically the average fidelity \bar{F} of the reconstructed state with respect to input state of the system qubit, for the case $T = 0$ and for the specific evolution $V(\phi, \theta = 0)$. Taking the excited state $|1\rangle$ as input, one has

$$\bar{F} \approx \frac{1}{n} + \frac{4[c/(1-c)]^2}{n(n-1)} \quad (18)$$

with $c = \cos \phi$, for n large enough; the expression for an arbitrary input state is Eq. (35). From these analytical results, one learns that the expected decrease of the average fidelity is much slower than the decrease in $1/n!$ of the probability of retrieving the initial state exactly.

3 Demonstration of the New Results

This is a technical section, devoted to the demonstration of the two new results. The common starting point is the form of the $(n+1)$ -qubit state obtained after interaction of the system qubit with n qubits in the bath. We consider an arbitrary pure input state $|\psi\rangle = c_0|0\rangle + c_1|1\rangle$; at $T = 0$, the equilibrium state is $\xi = P_0$ pure as well, so the initial state is $|\Psi_{in}\rangle = |\psi\rangle_S |0\dots 0\rangle_B$ and the state stays pure under the evolution. After the system has interacted with n qubits in the bath, the state reads

$$\mathcal{U}_n |\Psi_{in}\rangle = c_0 e^{in\theta} |0\rangle_S |0^n\rangle_B + c_1 \left[c^n |1\rangle_S |0^n\rangle_B + is |0\rangle_S \left(\sum_{k=1}^n c^{k-1} e^{i(n-k)\theta} |1_k\rangle_B \right) \right] \quad (19)$$

where $|0^n\rangle_B$ and $|1_k\rangle_B$ are n -qubit product states of all $|0\rangle$, respectively of $|1\rangle$ for qubit k and $|0\rangle$ for the others. Through all this section, we use $c = \cos \phi$, $s = \sin \phi$ and $B = \{1, \dots, n\}$.

3.1 Amount of multipartite entanglement

We want to compute the multipartite entanglement in the $(n+1)$ -qubit pure state (19). The measure of multipartite entanglement that we are going to compute is Eq. (6) of [13], Eq. (88) of [14]:

$$\mathcal{E}_{n+1} = 2^{1-\frac{n+1}{2}} \sqrt{2^{n+1} - 2 - S_{n+1}} \quad (20)$$

where S_{n+1} is the sum of $\text{Tr}(\rho_i^2)$ over all partial traces. Since the state is pure, the spectral properties of $\rho_{S,b}$ and of $\rho_{B \setminus b}$ are identical for all subset b of B ; therefore we must compute $S_{n+1} = 2 \sum_{b \neq B} \text{Tr} \rho_{S,b}^2$. Now, from (19) one has that all the $\rho_{S,b}$ are rank two states of the form $|\varphi_{S,b}\rangle \langle \varphi_{S,b}| + x_b |0\dots 0\rangle \langle 0\dots 0|$ with $|\varphi_{S,b}\rangle$ a non-normalized state whose precise form is not important here, and with

$$x_b = 1 - \|\varphi_{S,b}\|^2 = |c_1|^2 s^2 \sum_{k \notin b} c^{2(k-1)}. \quad (21)$$

So the quantity to be computed is

$$S_{n+1} = 2 \sum_{b \neq B} [(1 - x_b)^2 + x_b^2] = 2^{n+1} - 2 - 4 \left[\sum_{b \neq B} x_b - \sum_{b \neq B} x_b^2 \right]. \quad (22)$$

To compute the first sum, notice that $\frac{1}{|c_1|^2 s^2} \sum_{b \neq B} x_b = \sum_{b \neq B} \sum_{k \notin b} c^{2(k-1)} = \sum_{k=1}^n c^{2(k-1)} N_k$ where N_k is the number of sets b such that $k \notin b$. Clearly $N_k = 2^{n-1}$ for all k ; therefore we are left with a geometrical series and finally

$$\sum_{b \neq B} x_b = 2^{n-1} |c_1|^2 (1 - c^{2n}). \quad (23)$$

The second sum that we have to compute for (22) is

$$\sum_{b \neq B} x_b^2 = |c_1|^4 s^4 \left[\sum_{b \neq B} \sum_{k \notin b} c^{4(k-1)} + \sum_{b \neq B} \sum_{k, k' \notin b | k \neq k'} c^{2(k+k'-2)} \right] \equiv |c_1|^4 s^4 (I_1 + I_2). \quad (24)$$

Proceeding exactly as above, one obtains $I_1 = 2^{n-1} \frac{1-c^{4n}}{s^2(1+c^2)}$. Similarly,

$$I_2 = \sum_{k=1}^n \sum_{k' \neq k} c^{2(k+k'-2)} N_{kk'} = 2^{n-1} \frac{c^2(1-c^{2(n-1)})(1-c^{2n})}{s^4(1+c^2)} \quad (25)$$

because $N_{kk'} = 2^{n-2}$ for all k and $k' \neq k$, and we have applied to $x \equiv c^2$ the generic formula⁵

$$\sum_{k=0}^{n-1} \sum_{k' \neq k} x^{k+k'} = \frac{2x(1-x^{n-1})(1-x^n)}{(1-x)^2(1+x)}. \quad (26)$$

Inserting everything back into (22), then into (20), one obtains (17).

3.2 Irreversibility via random permutations in the reservoir

We start again from state (19). After the interaction with n qubits of the bath, we keep the system qubit and apply a random permutation $\pi : B \rightarrow B$ to the indices of the qubits in the bath. At this point, the state reads

$$\Pi \mathcal{U}_n |\Psi_{in}\rangle = c_0 e^{in\theta} |0\rangle_S |0^n\rangle_B + c_1 \left[c^n |1\rangle_S |0^n\rangle_B + i s |0\rangle_S \left(\sum_{k=1}^n c^{\tilde{\pi}(k)-1} e^{i(n-\tilde{\pi}(k))\theta} |1_k\rangle_B \right) \right] \quad (27)$$

with $\tilde{\pi} = \pi^{-1}$. On this state we apply the reverse evolution, at the end of which we find

$$\mathcal{U}_n^\dagger \Pi \mathcal{U}_n |\Psi_{in}\rangle = (c_0 |0\rangle + c_1 f_\pi |1\rangle)_S |0^n\rangle_B + c_1 \sqrt{1 - |f_\pi|^2} |0\rangle_S \sum_k \alpha_k |1_k\rangle_B \quad (28)$$

where we don't need the explicit form of the α_k , but we need

$$f_\pi = c^{2n} + s^2 \sum_{k=1}^n c^{k+\tilde{\pi}(k)-2} e^{i\theta[k-1-\tilde{\pi}(k-1)]}. \quad (29)$$

Let $\rho^{(n-, \pi, n-)} = \text{Tr}_B [\mathcal{U}_\pi |\psi, 0...0\rangle \langle \psi, 0...0| \mathcal{U}_\pi^\dagger]$ be the reconstructed state of the system qubit: its fidelity with respect to the initial state is

$$F(\pi) \equiv \langle \psi | \rho_S^{(n-, \pi, n-)} | \psi \rangle = |c_0|^2 + |c_1|^2 [|f_\pi|^2 + 2|c_0|^2 (\text{Re} f_\pi - |f_\pi|^2)]. \quad (30)$$

⁵ This formula can be re-derived from the usual sum of geometric series by noticing that $\sum_{k=0}^{n-1} \sum_{k' \neq k} x^{k+k'}$ is equal to $2 \sum_{k=0}^{n-2} \sum_{k'=k+1}^n x^{k+k'}$, or alternatively to $(\sum_k x^k)^2 - \sum_k x^{2k}$.

To quantify irreversibility, we must now average over all possible permutations. At this stage, we simplify the problem by focusing only on the unitary operation $V(\phi, \theta = 0)$, so that f_π becomes real⁶. To find $\bar{F} = \frac{1}{n!} \sum_\pi F(\pi)$ we have to compute

$$\bar{f}_\pi = c^{2n} + s^2 \frac{1}{n!} \sum_\pi \sum_{k=1}^n c^{k+\tilde{\pi}(k)-2} \equiv c^{2n} + s^2 I_1, \quad (31)$$

$$\bar{f}_\pi^2 = c^{4n} + 2c^{2n} s^2 I_1 + s^4 \frac{1}{n!} \sum_\pi \left(\sum_{k=1}^n c^{k+\tilde{\pi}(k)-2} \right)^2 \equiv c^{4n} + 2c^{2n} s^2 I_1 + s^4 I_2. \quad (32)$$

The pattern for these calculations is quite similar to the one we used in subsection 3.1. Consider first I_1 : observing that there are $(n-1)!$ permutations such that $\tilde{\pi}(k) = j$, one can decouple the two sums and write

$$I_1 = \frac{(n-1)!}{n!} \left(\sum_{k=1}^n c^{k-1} \right) \left(\sum_{j=1}^n c^{j-1} \right) = \frac{1}{n} \left(\frac{1-c^n}{1-c} \right)^2. \quad (33)$$

Similarly, using in particular (26), we obtain

$$\begin{aligned} I_2 &= \frac{1}{n!} \sum_\pi \sum_{k=1}^n c^{2(k+\tilde{\pi}(k)-2)} + \frac{1}{n!} \sum_\pi \sum_{k,k' | k \neq k'} c^{k+k'+\tilde{\pi}(k)+\tilde{\pi}(k')-4} \\ &= \frac{1}{n} \left(\frac{1-c^{2n}}{1-c^2} \right)^2 + \frac{1}{n(n-1)} \left(\frac{2c(1-c^{n-1})(1-c^n)}{(1-c)^2(1+c)} \right)^2. \end{aligned} \quad (34)$$

So finally the average fidelity for the reconstructed state is

$$\bar{F} = |c_0|^2 + |c_1|^2 [\bar{f}_\pi^2 + 2|c_0|^2(\bar{f}_\pi - \bar{f}_\pi^2)] \approx |c_0|^2 + |c_1|^2 [s^4 I_2 + 2|c_0|^2(s^2 I_1 - s^4 I_2)]. \quad (35)$$

Inserting the expressions above, one finds that the convergence $\bar{F} \rightarrow |c_0|^2$ is indeed present as expected, but is very slow (remember that $c = \cos \phi$ is close to 1 in the meaningful limit). The expression (18) given above is obtained by setting $c_0 = 0$ and $c^n \approx 0$.

4 Conclusions and Open Questions

In conclusion, we have reviewed a collisional model for the dynamical process of thermalization (or more generally, homogenization). In spite of its simplicity, the model exhibits a rich variety of interesting physical features, e.g. decoupling of dissipation and decoherence and exponential decays in time for both $(T_1$ and $T_2)$. Moreover, this kind of model provides a benchmark to study quantitative links between the parameters of thermalization and the entanglement generated by the evolution. This relation has been studied in this paper using a new measure of multipartite entanglement. Also as expected, if classical information (here, the labels of the particles in the bath) is lost after the entanglement has been distributed, the channel becomes irreversible. Here, this irreversibility has been quantified analytically for an example of channel.

Here is a list of open questions related to thermalization:

- Staying within the model, the two new results presented here are partial, and suggest themselves lines for further study. First of all, both results have been derived only for the case $T = 0$, because in this case the state of all the qubits is pure. In addition, the measure of multiparticle entanglement (20) has no clear-cut operational meaning⁷ and, as it turned out,

⁶ This is not the only simplification: if $\theta \neq 0$, expression (29) contains both $\tilde{\pi}(k-1)$ and $\tilde{\pi}(k)$.

⁷ It is known that this and similar measures of multipartite entanglement may detect entanglement beyond the bipartite case [15]. We note also that, contrary to what is stated in [9], the result obtained in [12] does not exclude the existence of multipartite entanglement in our thermalizing channels.

does not grasp all the physics of entanglement in the model, because the resulting quantity (17) is independent of θ . Similarly, the irreversibility has been studied only for the channel with $\theta = 0$, and in the case where only the bath qubits are permuted but the system qubit is known.

- The studied model has some remarkable features: e.g., the fact that the third requirement is automatically implied by the two others; the fact that (up to local unitaries) thermalizing channels are characterized by two parameters in one-to-one correspondence with the relaxation times T_1 and T_2 ; etc. Are these generic features of collisional thermalizing channels, or just artefacts of the choice of dealing with qubits?
- We discussed in the text the intuitive character of the partial swap. This operation is only possible because the system is of the same dimensionality as the particles in the bath. What happens if one wants to thermalize a d -dimensional quantum system with a bath of (say) qubits? Can the qubits be used one by one, or must one take them in bunches?

I am very grateful to the organizers of the 382. Wilhelm und Else Heraeus Seminar (Bad Honnef, 8-10.01.2007) for the invitation to attend this very stimulating workshop and give a talk there. This article benefits from discussions with Andreas Buchleitner, Daniel Burgarth, Oscar Dahlsten, Dai Li and Peter Zoller.

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